Rutgers University: Real Variables and Elementary Point-Set Topology Qualifying Exam

August 2016: Problem 2 Solution

Exercise. Let $a, b \in \mathbb{R}$ s.t. $a < b$ and let $f : [a, b] \to \mathbb{R}$

(a) Define what it means for f to be "absolutely continuous on $[a, b]$" (this is an $\epsilon - \delta$ definition).

Solution.

f is **absolutely continuous on** $[a, b]$ if $\forall \epsilon > 0$, $\exists \delta > 0$ s.t. for any finite set of disjoint intervals $(a_1, b_1), \ldots, (a_N, b_N)$ s.t. $(a_j, b_j) \subseteq [a, b]$ for all j,

$$\sum_1^N (b_j - a_j) < \delta \implies \sum_1^n |f(b_j) - f(a_j)| < \epsilon$$

(b) State a theorem relating the absolute continuity of such a function to its differentiability.

Solution.

The Fundamental Theorem of Calculus for Lebesgue Integrals:

If $-\infty < a < b < \infty$ and $f : [a, b] \to \mathbb{C}$, the following are equivalent

(a) f is absolutely continuous on $[a, b]$

(b) $f(x) - f(a) = \int_a^x f^*(t)dt$ for some $f^* \in L^1([a, b], m)$

(c) f is differentiable a.e. on $[a, b]$, $f' \in L^1([a, b], m)$ and $f(x) - f(a) = \int_a^x f'(t)dt$

(c) Assume that the restriction $f\Big|_{[\epsilon,1]}$ is absolutely continuous for every ϵ such that $0 < \epsilon < 1$, and that $\int_0^1 x^2|f'(x)|^p dx < \infty$ for some real number p s.t. $p > 3$. Prove that $\lim_{x\to 0} f(x)$ exists and is finite. (HINT: Prove that $\int_0^1 |f'(x)|dx < \infty$)

Solution.

If $f\Big|_{[\epsilon,1]}$ is absolutely continuous for every ϵ such that $0 < \epsilon < 1$, then f is differentiable a.e. on $[\epsilon, 1]$ for $0 < \epsilon < 1$

$\implies f$ is differentiable a.e. on $[0,1]$

Also, for $0 < \epsilon < 1$,

$$f(1) - f(\epsilon) = \int_\epsilon^1 f'(t)dt \qquad \implies \qquad f(\epsilon) = f(1) - \int_\epsilon^1 f'(t)dt$$

By the hint, we want to show $f' \in L^1([0,1])$.

$$\begin{aligned}
\int_0^1 |f'(x)|dx &= \int_0^1 |x^2(f'(x))^p|^{\frac{1}{p}}|x^{-\frac{2}{p}}|dx \\
&= \int_0^1 x^{\frac{2}{p}}|f'(x)| \cdot x^{-\frac{2}{p}}dx \\
&\leq \left[\int_0^1 \left|x^{\frac{2}{p}}f'(x)\right|^p dx\right]^{\frac{1}{p}} \cdot \left[\int_0^1 x^{-\frac{2}{p}\cdot\frac{p}{p-1}}dx\right]^{\frac{p-1}{p}} \qquad \text{by Holders inequality} \\
&= \left[\int_0^1 x^2 \left|f'(x)\right|^p dx\right]^{\frac{1}{p}} \cdot \left[\int_0^1 x^{-\frac{2}{p-1}}dx\right]^{\frac{p-1}{p}}
\end{aligned}$$

$$\left[\int_0^1 x^2 \left|f'(x)\right|^p dx\right]^{\frac{1}{p}} < \infty \qquad \text{and} \qquad p > 3 \implies p-1 > 2 \implies 1 > \frac{2}{p-1}$$

$$\implies \qquad \int_0^1 x^{-\frac{2}{p-1}}dx < \infty \qquad \text{since} \qquad x^{-\frac{2}{p-1}} < \infty \text{ is integrable for } x \in [0,1]$$

$$\implies \qquad \lim_{x\to 0} f(x) = \lim_{x\to 0}\left(f(1) - \int_x^1 f'(u)du\right) = f(1) - \int_0^1 f'(u)du < \infty$$